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Sample-to-Sample Fluctuations in the Conductivity of a Disordered Medium

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We investigate the sample-to-sample fluctuations in the conductivity of a random resistor network—equivalently, in the diffusivity of a disordered medium with symmetric hopping rates. We argue that whenever the effective conductivity σ^* is strictly positive, then the fluctuations are normal, i.e., proportional to $(\text{volume})^{-1/2}$. If the local conductivities are allowed to be zero, then σ^* vanishes when approaching the percolation threshold p_c. Close to p_c the fluctuations are anomalous. From the renormalization group on hierarchical lattices we find that at p_c fluctuations and mean scale in the same fashion, i.e., there is no independent scaling exponent for the fluctuations.

KEY WORDS: Random resistor network; percolation threshold; scaling exponents.

1. INTRODUCTION

It has been recognized for some time that sample-to-sample fluctuations are an important issue for disordered systems. If the quantity of physical interest is not self-averaging as the size of the system becomes large, then a mere disorder average may be meaningless and the full distribution must be elucidated. But even for quantities which are self-averaging the fluctuations (with respect to disorder) carry information on qualitative changes in the behavior of the system. For example, the free energy of the Sherrington–Kirkpatrick model of a spin glass is self-averaging, but fluctuations change when passing through T_c.[1]

In this paper we investigate the sample-to-sample fluctuations in the conductivity of a random resistor network (equivalently in the diffusivity

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for a random walk in a symmetric random medium). The model is standard: To each bond b of a regular lattice we assign a random conductance σ_b. The conductances are assumed to be statistically independent. A sample with volume Λ has then the conductivity σ_Λ. The precise definition will be explained in Section 2. It is known that $\sigma_\Lambda \to \sigma^*$ as $|\Lambda| \to \infty$ for almost every realization of the resistor network.[2] σ^* is the macroscopic (effective) conductivity of the medium. σ^* does not fluctuate. Hence σ_Λ is self-averaging. The problem under consideration is to determine the fluctuations of σ_Λ around σ^*.

The one-dimensional case is worked out easily, since the total resistance is the sum of single-bond resistors. Therefore one just has to add independent random variables. σ_Λ has Gaussian fluctuations of the order $|\Lambda|^{-1/2}$ around σ^* provided small values of σ_b are sufficiently unlikely. In higher dimensions the current has more possibilities to cross the sample and the conductivity is no longer a sum of independent random variables. Still, as will be supported by a perturbational calculation in Section 2, σ_Λ has normal fluctuations provided $\sigma^* > 0$. Thus, interesting fluctuation behavior can occur only close to critical points where σ^* vanishes. Random resistor networks with such a transition have been studied in great detail; see refs. 3 and 4 for recent reviews. One assumes that $\sigma_b = 0$ with probability $1 - p$ and $\sigma_b > 0$ with probability p. In the case of a random mixture of conducting and insulating material one would choose $\sigma_b = 1$ with probability p. If p is close to one, then the conducting bonds percolate throughout the sample and $\sigma^* > 0$. As p is decreased to the percolation threshold p_c, the effective conductivity vanishes continuously. To understand the fluctuation behavior for $p > p_c$ but close to p_c it is convenient to introduce the percolation correlation length ξ. For length scales larger than ξ the network looks like a regular lattice with small fluctuations in the conductivity, whereas for distances less than ξ there are strong fluctuations combined with an irregular local connectivity. Thus, for a sample of linear size L, $L \ll \xi$, we expect to have fluctuations in the conductivity of the form

$$\langle \sigma_\Lambda^2 \rangle - \langle \sigma_\Lambda \rangle^2 \sim (\xi/L)^{-d/2} \tag{1.1}$$

Of course, the whole problem is to understand how the prefactor on the right side of (1.1) scales as p_c is approached. This will be the main focus of our investigation.

2. Conductivity and Weak-Disorder Expansion

In the usual experimental setup one measures the current across the sample with a unit voltage drop maintained externally. We consider then a slab Λ of the d-dimensional lattice Z^d and write $x = (x_1, x_\perp) \in Z^d$. Inde-

pendently, to each bond $b=(x, y)$, $|x-y|=1$, we attach the conductance $\sigma_{(x,y)}=\sigma_{(y,x)}\geqslant 0$. At the right and left boundaries of Λ, $\{x_1=0\}$, $\{x_1=L\}$, the voltage ϕ is prescribed. Λ has periodic boundary conditions along $x_\perp$. Let us define the linear operator

$$Af(x)=\sum_{y,|x-y|=1}\sigma_{(x,y)}[f(y)-f(x)] \tag{2.1}$$

The steady-state voltage then satisfies

$$A\phi(x)=0 \tag{2.2}$$

with $\phi(0, x_\perp)=L$, $\phi(L, x_\perp)=0$. The current through the bond (x, y) is given by

$$j_{(x,y)}=-\sigma_{(x,y)}(\phi(y)-\phi(x)) \tag{2.3}$$

We set $\phi(x)=L-x_1+\delta\phi(x)$. Inserting in (2.2) and solving for $\delta\phi$, one obtains the steady-state current and thereby the conductivity along the one-axis as

$$\sigma_\Lambda=|\Lambda|^{-1}\sum_{x\in\Lambda}\{\sigma_{(x,x+e_1)}+u_1(x)\,A^{-1}u_1(x)\} \tag{2.4}$$

Here

$$u_\alpha(x)=\sum_{e,|e|=1}\sigma_{(x,x+e)}e_\alpha \tag{2.5}$$

with e_α the α th unit vector. A is to be understood with *zero* boundary conditions at $\{x_1=0\}$, $\{x_1=L\}$ and periodic boundary conditions otherwise.

Since for a given realization of bond conductances the resistor network is inhomogeneous, the effective conductivity is really a $d\times d$ tensor. In (2.4) we computed only its (11) component. The tensorial character is more easily displayed for a (slightly unphysical) sample with periodic boundary conditions. We define the conductivity through the response in the current to a weak electric driving field. Then

$$(\sigma_\Lambda)_{\alpha\beta}=|\Lambda|^{-1}\sum_{x\in\Lambda}\left\{\tfrac{1}{2}\sum_{e,|e|=1}\sigma_{(x,x+e)}e_\alpha e_\beta+u_\alpha(x)\,A^{-1}u_\beta(x)\right\} \tag{2.6}$$

Here A is understood with periodic boundary conditions.

For the weak-disorder expansion we use (2.6) as starting point. (For the average conductivity a weak-disorder expansion has been carried out in ref. 5 and an expansion in the impurity concentration in ref. 6.) We

expand around the uniform medium with conductivity $\sigma_0 = \langle \sigma_b \rangle$. Then $\sigma_{(x,y)} = \sigma_0 + \delta\sigma_{(x,y)}$, where $\langle \delta\sigma_{(x,y)} \rangle = 0$,

$$\langle \delta\sigma_{(x,x')} \delta\sigma_{(y,y')} \rangle = (\delta_{xy}\delta_{x'y'} + \delta_{xy'}\delta_{x'y}) \langle (\sigma_b - \langle \sigma_b \rangle)^2 \rangle$$

It is assumed here that $\langle (\sigma_b - \langle \sigma_b \rangle)^2 \rangle^{1/2} = \varepsilon \ll \sigma_0$. Correspondingly, A and u split into a constant and a fluctuating part, $A = A_0 + \delta A$ and $u = u_0 + \delta u$, where $u_0 = 0$ by (2.5). Using the Dyson series for A^{-1}, we obtain

$$\begin{aligned}(\sigma_A)_{\alpha\beta} = |A|^{-1} \sum_{x \in A} \Bigg\{ & \delta_{\alpha\beta}\sigma_0 + \sum_{e, |e|=1} \delta\sigma_{(x, x+e_\alpha)} + \sum_y \delta u_\alpha(x)\, G(x, y)\, \delta u_\beta(y) \\ & - \sum_{y,z} \sum_{e, |e|=1} \delta u_\alpha(x)\, G(x, y)\, \delta\sigma_{(y, y+e)} \\ & \times (G(y+e, z) - G(y, z))\, \delta u_\beta(z) + \cdots \Bigg\} \end{aligned} \tag{2.7}$$

Here G is the inverse of A_0 and is given by

$$G(x, y) = |A|^{-1} \sum_{k \in \mathrm{BZ}} e^{ik(x-y)} \varepsilon(k)^{-1}, \qquad \varepsilon(k) = \sigma_0 \sum_{\alpha=1}^{d} 2(1 - \cos k_\alpha) \tag{2.8}$$

where the sum is over the first Brillouin zone.

We have to compute the variance $\langle (\sigma_A)^2_{\alpha\beta} \rangle - \langle (\sigma_A)_{\alpha\beta} \rangle^2$. To order ε^2 only the first term in (2.7) contributes. The second term, $\delta u A_0^{-1} \delta u$, gives a contribution of order ε^4. Note that the cross terms between the first and second summands vanish. To order ε^4 we have, then, in the case of three dimensions,

$$\begin{aligned} \langle (\sigma_A)^2_{\alpha\beta} \rangle - \langle (\sigma_A)_{\alpha\beta} \rangle^2 &= |A|^{-1} \{ \delta_{\alpha\beta}\varepsilon^2 + \sigma_0^{-2} (\tfrac{1}{3}\delta_{\alpha\beta}\varepsilon^4 + \tfrac{1}{9}\delta_{\alpha\beta}\bar{\varepsilon}^4 + \tfrac{1}{12}\varepsilon^4) \} \\ \bar{\varepsilon}^4 &= \langle \sigma_b^4 \rangle - \langle \sigma_b^2 \rangle^2 \end{aligned} \tag{2.9}$$

To this order, the fluctuations in the conductivity are proportional to $|A|^{-1/2}$.

To be sure that the fluctuating part of A does not produce anomalous behavior, we also computed the contribution of the last term in (2.7), which generates a term of order ε^6 for the variance. Again all Brillouin zone integrals converge and the variance is proportional to the inverse of the volume. We expect this behavior to hold to all orders.

3. REAL-SPACE RENORMALIZATION HIERARCHICAL LATTICES

To understand the fluctuations in the conductivity near the percolation threshold, one strategy would be to follow the change in the distribution of

σ_A as the system size is doubled. Such an exact renormalization is difficult to handle, unfortunately, and one is forced to use approximations. In our context such an approximate renormalization group was first introduced by Stinchcombe and Watson,[8] who used it to determine the average conductivity near threshold. The Stinchcombe–Watson renormalization can be interpreted as an exact renormalization on some hierarchical lattice.[7] We will adopt this point of view. The critical behavior on a hierarchical lattice exhibits in general nonclassical critical exponents which depend on the connectivity of the lattice. However, one lacks a precise linkage between the dimension of the regular lattice and the connectivity of the approximating hierarchical lattices. Thus quantitative predictions should be taken with a grain of salt.

In the following we examine two specific examples approximating two-dimensional regular lattices.[8,9] The general case will be discussed in Section 6.

The first model is the standard hierarchical lattice with branching ratio two (Fig. 1). To the bonds in the second graph one assigns the conductivities $\sigma_1,\ldots,\sigma_4$. Clearly the conductance between A and B is then

$$\sigma' = (\sigma_1^{-1} + \sigma_2^{-1})^{-1} + (\sigma_3^{-1} + \sigma_4^{-1})^{-1} \tag{3.1}$$

Now, if we assume that $\sigma_1,\ldots,\sigma_4$ are independent random variables with common distribution $\mu(d\sigma)$, then "=" in (3.1) should be read as an equality for distributions and the meaning of Eq. (3.1) is to give the distribution of the conductivity between A and B. More explicitly, if $\mu_n(d\sigma)$ is the distribution of the conductance at level n, then at level $n+1$

$$\mu_{n+1}(d\sigma) = \int \mu_n(d\sigma_1)\,\mu_n(d\sigma_2)\,\mu_n(d\sigma_3)\,\mu_n(d\sigma_4) \times \delta(\sigma - [(\sigma_1^{-1} + \sigma_2^{-1})^{-1} + (\sigma_3^{-1} + \sigma_4^{-1})^{-1}]) \tag{3.2}$$

We want to avoid such cumbersome notation and will use = for equality between distributions. The large-size properties of the conductance are obtained by iterating the recursion (3.2) many times.

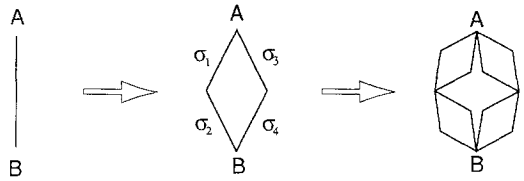

Fig. 1. Construction of a hierarchical lattice with branching ratio two.

Another proposal[9] is the Wheatstone bridge (Fig. 2). Its recursion reads

$$\sigma' \triangleq \frac{\sigma_3[\sigma_1(\sigma_2+\sigma_4+\sigma_5)+\sigma_2\sigma_5]+\sigma_4[\sigma_2(\sigma_1+\sigma_3+\sigma_5)+\sigma_1\sigma_5]}{(\sigma_1+\sigma_3)(\sigma_2+\sigma_4)+\sigma_5(\sigma_1+\sigma_2+\sigma_3+\sigma_4)} \tag{3.3}$$

It is claimed that the recursion (3.3) is a better approximation to the two-dimensional lattice than (3.1).

In the following section we will investigate the flow on the space of measures defined by (3.2). The discussion of (3.3) is completely analogous. We studied both (3.2) and (3.3) by means of Monte Carlo renormalization. The outcome for (3.3) is discussed in Section 5.

4. FLOW

The iteration (3.2) remains meaningful if we allow $\sigma_1,\ldots,\sigma_4$ to vanish. Therefore $\mu_n(d\sigma)$ is concentrated on the closed half-line $[0,\infty)$. We first note that any constant value σ^*, $\sigma^* \geqslant 0$, is a fixed point for (3.2). Thus we expect that a given starting distribution $\mu_0(d\sigma)$ will flow under (3.2) to $\delta(\sigma-\sigma^*)$ with some σ^*, which is then the effective conductivity of the hierarchical lattice with single-bond conductivity $\mu_0(d\sigma)$. To understand the geometry of the flow and its critical behavior, it is useful to decompose μ_n as

$$\mu_n(d\sigma) = (1-p_n)\,\delta(\sigma) + p_n\mu'(d\sigma), \qquad \int_0^\infty \mu'(d\sigma) = 1 \tag{4.1}$$

and μ' concentrated on the open half-line $(0,\infty)$. Substituting (4.1) in (3.2), we obtain the iteration

$$p_{n+1} = f(p_n) = 2p_n^2 - p_n^4 \tag{4.2}$$

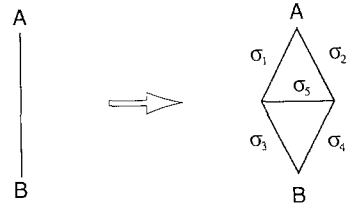

Fig. 2. Wheatstone hierarchical lattice.

which determines the weight at zero and

$$\sigma^{(n+1)} \hat{=} [(\sigma_1^{(n)})^{-1} + (\sigma_2^{(n)})^{-1}]^{-1} + [(\sigma_3^{(n)})^{-1} + (\sigma_4^{(n)})^{-1}]^{-1}$$
$$\text{with probability} \quad p_n^4/(2p_n^2 - p_n^4) \tag{4.3}$$

$$\sigma^{(n+1)} \hat{=} [(\sigma_1^{(n)})^{-1} + (\sigma_2^{(n)})^{-1}]^{-1}$$
$$\text{with probability} \quad 2p_n^2(1-p_n^2)/(2p_n^2 - p_n^4) \tag{4.4}$$

for the remainder of the distribution. Here $\sigma_i^{(n)} > 0$, $i = 1,..., 4$.

We observe that the one-dimensional map (4.2) has a single repulsive fixed point at $p = p_c$ defined through $f(p_c) = p_c$, $p_c \simeq 0.62$, and two attractive fixed points at $p = 0, 1$. If for the starting distribution $p > p_c$, then p flows to 1 under (4.2) and the remainder of the distribution follows (4.3). Since there $\sigma^{(n)} > 0$, we expect a limit distribution, concentrated at some $\sigma^* > 0$. To see how $\mu_n'(d\sigma)$ approximates its limit, we linearize (4.3) as $\sigma^* + \delta\sigma_i^{(n)}$. This yields

$$\delta\sigma^{(n+1)} \hat{=} \frac{1}{4} \sum_{i=1}^{4} \delta\sigma_i^{(n)} \tag{4.5}$$

Therefore

$$\langle (\delta\sigma^{(n)})^2 \rangle^{1/2} \sim 2^{-n} \tag{4.6}$$

and $\delta\sigma^{(n)}$ has a Gaussian distribution on that scale, provided the second moment of $\delta\sigma^{(n)}$ is finite. The number of bonds increases as 4^n. The effective conductivity is of the order of 1. Thus (4.6) means that the fluctuations of the conductance are normal and have a size of the order $(\text{volume})^{-1/2}$. (For a particular hierarchical lattice Gaussian fluctuations are proved in ref. 10.) We emphasize that the expansion (4.5) makes sense only if $\sigma^* > 0$.

On the other hand, if for the initial distribution $p < p_c$, then p flows to zero under (4.2). The effective conductivity vanishes. The rest of the distribution is governed by (4.4), which corresponds to resistors in sequence. If $\langle (\sigma^{(0)})^{-2} \rangle < \infty$, then according to (4.4), $\langle \sigma^{(n)} \rangle \sim 2^{-n}$ and $\langle (\sigma^{(n)} - \langle \sigma^{(n)} \rangle)^2 \rangle^{1/2} \sim (2\sqrt{2})^{-n}$ with Gaussian fluctuations on that scale.

The only case left is a starting distribution with a weight at zero given exactly by $1 - p_c$. The weight at zero does not change then and the remainder flows under a nondegenerate combination of the two iterations (4.3) and (4.4). We could clarify this scaling only through Monte Carlo renormalization. In comparison to other RG flows, the simplifying feature of random conductivities on a hierarchical lattice is that the critical manifold is determined already by the one-dimensional map (4.2).

5. MONTE CARLO RENORMALIZATION

The only feasible method for a numerical investigation of the iteration (3.3) seems to be a Monte Carlo procedure. The basic algorithm is simple. If the starting distribution is binary, $\mu^{(0)}(d\sigma) = (1-p)\,\delta(\sigma) + p\delta(\sigma-1)$, then one generates a large array of independent random numbers with that distribution. An iterated array is generated by calculating out of five random numbers a new one according the rule (3.3). Since in each step the size of the array decreases by a factor of five, after a few iterations a rather small array is left with inherently large statistical errors. To avoid this difficulty, we give up strict independence and reduce in each iteration the size of the array only by a factor of 1.3 according to some random algorithm. In this fashion we can iterate up to 30 times, keeping statistical errors on a reasonable level. For particular cases we checked the stability of this procedure against runs with strict independence.

Figure 3a shows the effective conductivity σ^* versus p for a binary distribution on a Wheatstone hierarchical lattice. The percolation threshold is at $p_c = 0.5$. As expected, the average conductivity is almost linear and vanishes as $(p-p_c)^t$ at p_c. The critical region is too small for a reliable determination of the exponent t. From the scaling relation (7.5) one infers that $t \cong 1.3$. In Fig. 3b we show the width of the distribution for σ after 18 renormalization steps. If p is not too close to p_c, then the width scales as 2^{-n}, in agreement with the linearization (4.5). Close to p_c it takes more steps to reach the scaling regime, a behavior to be discussed in more detail in Section 7.

This leaves us with the critical cases. Fortunately the iteration for p alone determines already the unstable manifold of Eq. (4.2), which implies that for any choice of p, $p \neq 0, 1$, in (4.3) and (4.4) the flow should be critical. This extra parameter in (4.3) and (4.4) is most welcome as a check on numerical stability. Of course, the scaling exponents of interest have to be computed with $p = p_c$. We iterated according to (4.3) with probability w and according to (4.4) with probability $1-w$, $0 < w < 1$. The invariant distribution was reached already after 3–4 steps. It then scaled self-similarly as

$$\mu^{(n+1)}(d\sigma) = \mu^{(n)}(\lambda\, d\sigma) \tag{5.1}$$

To a good approximation the scale factor λ follows the linear dependence $\lambda(w) = w + 2(1-w)$. We recall that for the iteration (4.3), i.e., $w = 1$, $\langle \sigma^{(n)} \rangle = O(1)$, whereas for the iteration (4.4), i.e., $w = 0$, $\langle \sigma^{(n)} \rangle = 2^{-n}$. Thus, both limiting behaviors are reproduced and the interpolation is as simple as possible. The fixed point distribution seems to be very robust against small numerical errors. In Fig. 4 we show the invariant distribution

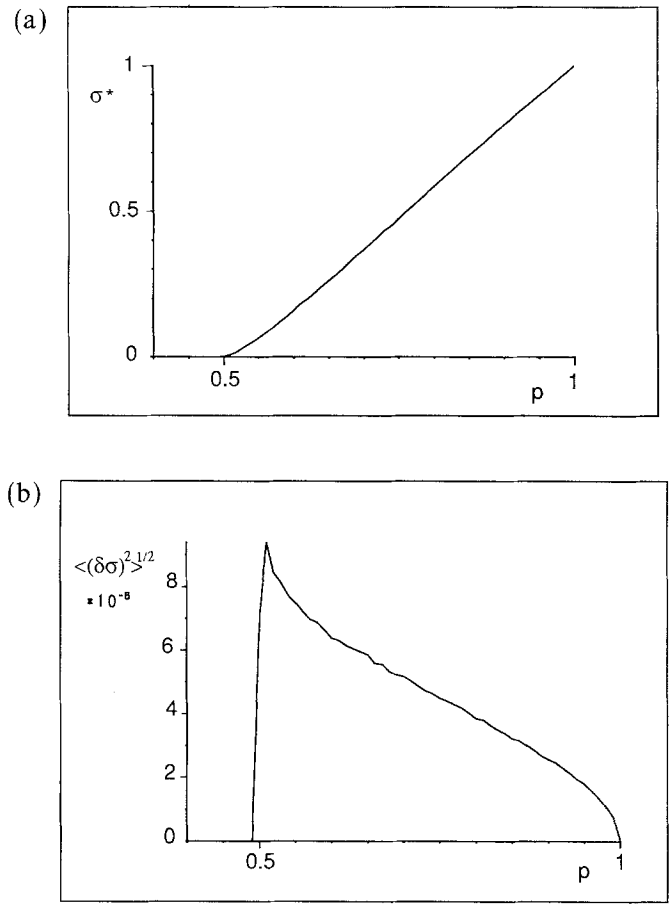

Fig. 3. (a) Average conductivity for a binary distribution on a Wheatstone hierarchical lattice. (b) Standard deviation in the conductivity for a binary distribution on a Wheatstone hierarchical lattice.

for the Wheatstone iteration at $p = p_c = 0.5$. The scale factor is $\lambda \cong 1.9$. For the iteration (3.2) at p_c the scale factor is $\lambda \cong 1.76$.

The Monte Carlo renormalization reveals a rather unexpected feature. At p_c there is only a single length scale in the sense that the mean and width of the distribution for the conductivity scale in precisely the same fashion. Since this property holds for various hierarchical lattices, we believe it not to be accidental and conjecture the same property for regular lattices. If so, then at $p = p_c$, $\langle \sigma_A \rangle$ and $\langle (\sigma_A - \langle \sigma_A \rangle)^2 \rangle^{1/2}$ have to vanish at the same rate as $|A| \to \infty$.

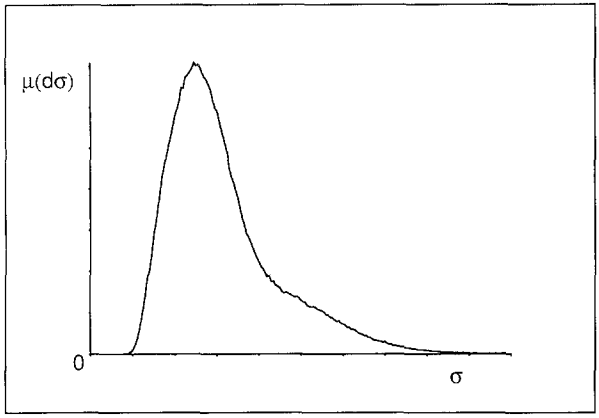

Fig. 4. Invariant distribution for the Wheatstone hierarchical lattice at $p_c = 0.5$ (arbitrary scale).

6. GENERAL HIERARCHICAL LATTICES

In general, the substitution at a single bond could have an arbitrary geometric form. As an analogue of spatial dimension one commonly introduces a hierarchical lattice with z branches instead of two as in Fig. 1. But other substitutions have also been proposed in order to have an improved approximation to regular lattices.

Given the substitution rule, we have a recursion of the form (3.1), (3.3). For these two models $\sigma^{(n)}$ is of the order 1. In general, this is not the case and we have to rescale the conductivity as $\sigma'^{(n)} = a^n \sigma^{(n)}$. The rescaling factor a is determined through the ordered lattice, i.e., with all bond conductivities equal to 1. In terms of $\sigma'^{(n)}$ the previously developed picture of the renormalization group flow is not altered.

The recursion for $\sigma'^{(n)}$ has as fixed points the constants σ^*, $\sigma^* \geqslant 0$. If $p > p_c$, then $\sigma^* > 0$ and we are allowed to linearize around σ^*. This determines the fluctuations as

$$\langle (\delta\sigma^{(n)})^2 \rangle^{1/2} \sim b^{-n} \tag{6.1}$$

for large n with b computed from the linearization. For a regular lattice the fluctuations in the conductivity are of the order $L^{-d/2}$ with L the linear dimension of the box. We identify L with the shortest path in the hierarchical lattice. If for the basic cell the shortest path has q bonds, then $L = q^n$. Thus, it is natural to define the dimension of the hierarchical lattice as

$$d = 2 \frac{\log(b)}{\log(q)} \tag{6.2}$$

In the cases (3.1), (3.3) we have $d=2$, and for a branching ratio z hierarchical lattice

$$d=\frac{\log(2z)}{\log(2)} \tag{6.3}$$

At p_c we obtain, as before, a mixture of recursions which determines the fixed point distribution on a single length scale. There is one additional simplifying feature. For high dimensions, p_c is very close to zero. Therefore the recursions have vastly different weights. For example, for the branching ratio $z=3$ (4) hierarchical lattice, $p_c=0.389$ (0.282). The two-bond recursion has the largest probability of 0.327 (0.243) followed by the four-bond recursion with probability 0.059 (0.029). Since the scaling exponent seems to depend rather smoothly on the parameters of the mixture, a natural approximation is to keep only the highest-weight recursion. Of course, the mixture is needed in order to have a single length scale. Applying this approximation to the branching ratio z hierarchical lattice, we obtain $\lambda=3$, compared to the numerical value of $\lambda=2.72$ for $z=3$ and $\lambda=4$ compared to $\lambda=3.75$ for $z=4$.

7. SCALING BEHAVIOR

We return now to regular d-dimensional lattices and develop a scaling theory for the fluctuations near p_c on the basis of our results from the hierarchical lattices. Let us first recall the behavior of the average conductivity. The sample is a box A with linear dimension L and volume L^d. Close to p_c the effective conductivity σ^* scales as

$$\sigma^*=\lim_{|A|\to\infty}\langle\sigma_A\rangle\sim(p-p_c)^t,\ p>p_c \tag{7.1}$$

Exactly at p_c the average conductivity vanishes with the linear size of the system as

$$\langle\sigma_A\rangle|_{p_c}\sim L^{-\mu} \tag{7.2}$$

To relate t and μ, we argue that for p close to p_c one sees a behavior like (7.2) up to a length of the order ξ, with ξ the correlation length of the percolation. Now

$$\xi\sim(p-p_c)^{-\nu} \tag{7.3}$$

and therefore

$$\langle\sigma_A(p)\rangle\sim\xi^{-\mu}\sim[(p-p_c)^{-\nu}]^{-\mu}=(p-p_c)^t \tag{7.4}$$

which implies

$$t = \nu\mu \tag{7.5}$$

For the sample-to-sample fluctuations it is natural to introduce in the same spirit also two scaling exponents. For p close to p_c we have

$$(\langle \sigma_A^2 \rangle - \langle \sigma_A \rangle^2)^{1/2} \sim (p - p_c)^{-\gamma} L^{-d/2} \tag{7.6}$$

where we anticipate the anomalous behavior of the amplitude. Exactly at p_c the fluctuations vanish as

$$(\langle \sigma_A^2 \rangle - \langle \sigma_A \rangle^2)^{1/2}|_{p_c} \sim L^{-\beta} \tag{7.7}$$

Now for the hierarchical lattice we observed that at p_c the mean and width scale in the same fashion. If this remains to be so for regular lattices, then

$$\beta = \mu \tag{7.8}$$

As before we argue that the behavior (7.7) is seen in (7.6) up to a length ξ. Equating coefficients yields

$$\xi^{-2\beta} = (p - p_c)^{-2\gamma} \xi^{-d} \tag{7.9}$$

and therefore

$$\gamma = \tfrac{1}{2}\nu(d - 2\beta) \tag{7.10}$$

Table I

	ν	t	μ	γ
Hierarchical lattice[a]				
Branching ratio 2	1.62	1.30	0.84	0.52
Wheatstone	1.43	1.32	0.93	0.2
				0.27 ± 0.07
Regular lattice[b]				
$d = 2$	4/3	1.3	0.97	0.08
$d = 3$	0.88	2.02	2.30	−1.4
$d = 6$	1/2	3	6	−3

[a] ν is computed from the linearization of (4.2). μ is determined from the MC renormalization. t and γ are obtained from the scaling relations (7.5) and (7.10). For the Wheatstone lattice γ is compared with the numerical value obtained from Fig. 4.

[b] The values for ν and t are taken from ref. 3. μ and γ are obtained through the scaling relations (7.5) and (7.10).

Thus the scaling behavior of the fluctuations can be summarized as

$$(\langle \sigma_A^2 \rangle - \langle \sigma_A \rangle^2)^{1/2} \sim \langle \sigma_A \rangle (\xi/L)^{-d/2} \tag{7.11}$$

for $L \gg \xi$. The relation (7.11) identifies the missing prefactor in (1.1) and exploits the fact that close to p_c the conductivity is governed by a *single* independent scaling exponent.

For the convenience of the reader we list in Table I the numerical values for the various exponents and indicate how they have been obtained. $d=6$ is the upper critical dimension.[3] Note that by (7.6) and (7.11) the numerical value of γ results in a competition between the vanishing of $\langle \sigma_A \rangle$ and the increase of the correlation length ξ. For $d=2$ the second term barely wins and $\gamma > 0$, whereas for $d \geqslant 2$ negative, tending to the mean field value $\gamma = -3$.

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REFERENCES

1. M. Aizenman, J. L. Lebowitz, and D. Ruelle, *Commun. Math. Phys.* **112**:3 (1987).
2. K. Golden and G. C. Papanicolaou, *Commun. Math. Phys.* **90**:473 (1983).
3. J. P. Bouchaud and A. Georges, *Phys. Rep.* **195**(4&5):127 (1990).
4. S. Havlin and D. Ben Avraham, *Adv. Phys.* **36**:695 (1987).
5. B. Derrida and J. M. Luck, *Phys. Rev. B* **28**:7183 (1983).
6. M. H. Ernst and R. F. J. van Velthoven, *J. Stat. Phys.* **45**:1001 (1986).
7. A. N. Berker and S. Ostlund, *J. Phys. C* **12**:4961 (1979).
8. R. B. Stinchcombe and B. P. Watson, *J. Phys. C* **9**:3221 (1976).
9. J. Bernasconi, *Phys. Rev. B* **18**:2185 (1978).
10. C. D. Essoh and J. Bellissard, *J. Phys. A* **22**:4537 (1989).